Through the use of the well-known Morrison/Wheat/Erkut ice hockey model, we demonstrate that even optimal solutions may be open to question. Based on the assumptions which are made to estimate data values, optimal solutions may vary among decision makers. We also note that under any reasonable set of assumptions, hockey coaches wait too long before pulling their goalies.

In ancient times, Morrison [1976] presented the seminal analysis on the optimal time to pull a goalie in an ice hockey game. More recently, Morrison and Wheat [1986] realized that the original paper contained a major error and published a revised, corrected paper. Most recently, Erkut [1987] generalized the Morrison/Wheat (MW) model so that his students would not feel "uneasy" about the Morrison and Wheat assumption that essentially lumped Erkut's NHL Champion Edmonton Oilers with Morrison's beloved and improved Detroit Red Wings. We, too, feel that the generalization made by Erkut is a major contribution to ice hockey as we dread the lumping of our Broad Street Bullies with other teams. While we very much endorse the generalization of the formula for the optimal time to pull the goalie as given by

\[ t^* = \frac{\ln((-L_1L_B + L_2L_A)/L_A(L_1 + L_2 - L_A - L_B))}{L_A + L_B}, \]

where

- \( L_1 \) and \( L_2 \) are the even strength scoring rates of teams 1 and 2,
NYDICK, WEISS

$L_A$ is the scoring rate for team 1 when it has pulled its goalie, and $L_H$ is the scoring rate for team 2 when team 1 has pulled its goalie, we note an alternative interpretation of this equation.

The formula requires the estimation of the scoring rates of each team at even strength ($L_1$ and $L_2$) and the scoring rates when team 1 has an extra skater because it pulled its goalie ($L_A$ and $L_H$). The Morrison/Wheat formula has only one even-strength scoring rate, $L (= L_1 = L_2)$. MW used actual NHL data to estimate $L$, $L_A$, and $L_H$, as 0.06, 0.16, and 0.47 goals per minute respectively. (Note that MW assumed that all teams score at the same rate when playing at even strength.) Using individual team data from the NHL, Erkut found that the even strength scoring rates vary by team from 0.05 to 0.09 goals per minute. To estimate $L_A$ and $L_H$, he argued that, according to the MW data, the scoring rate increases by a factor (multiple) of 2.67 when a team pulls its goalie and that the opponent's scoring rate increases by a factor of 7.83. Using $L_A = 2.67L_1$ and $L_H = 7.83L_2$, Erkut generated the optimal time to pull the goalie for various pairs of $L_1$ and $L_2$ as shown in Table 1.

Erkut's assumption implies that pulled-goalie scoring rates are directly proportional to the even-strength scoring rates. Alternatively, the pulled-goalie scoring rates could be estimated differently. Since MW estimated that $L_A = 0.16$ and $L_H = 0.47$, constant uneven strength scoring rates could be used regardless of which teams are playing. This fixed rate assumption contradicts the spirit of the generalization developed by Erkut, but we feel that the true behavior of the scoring rates when there is a one skater advantage is actually somewhere between Erkut's proportionality assumption and the fixed rate assumption. Ideally, if the model is robust, then modifying the

\begin{tabular}{lccccc}
\hline
$L_2$ & 0.05 & 0.06 & 0.07 & 0.08 & 0.09 \\
\hline
0.05 & 2.82 & 2.40 & 2.09 & 1.85 & 1.66 \\
0.06 & 2.76 & 2.35 & 2.05 & 1.82 & 1.63 \\
0.07 & 2.69 & 2.30 & 2.02 & 1.79 & 1.61 \\
0.08 & 2.63 & 2.26 & 1.98 & 1.76 & 1.59 \\
0.09 & 2.58 & 2.22 & 1.95 & 1.74 & 1.57 \\
\hline
\end{tabular}

Table 1: Optimal time to pull the goalie assuming $L_A = 2.67L_1$ and $L_H = 7.83L_2$.

\begin{tabular}{lccccc}
\hline
$L_2$ & 0.05 & 0.06 & 0.07 & 0.08 & 0.09 \\
\hline
0.05 & 2.70 & 2.84 & 3.00 & 3.19 & 3.42 \\
0.06 & 2.25 & 2.35 & 2.46 & 2.58 & 2.72 \\
0.07 & 1.88 & 1.96 & 2.04 & 2.13 & 2.23 \\
0.08 & 1.58 & 1.63 & 1.69 & 1.76 & 1.83 \\
0.09 & 1.31 & 1.36 & 1.40 & 1.45 & 1.50 \\
\hline
\end{tabular}

Table 2: Optimal time to pull the goalie assuming $L_1 = 0.16$ goals/minute and $L_H = 0.47$ goals/minute.
PULLING THE GOALIE

<table>
<thead>
<tr>
<th>Scoring rates</th>
<th>Proportionality assumption</th>
<th>Fixed rate assumption</th>
</tr>
</thead>
<tbody>
<tr>
<td>( L_1 = 0.05 )</td>
<td>( i^* = 1.66 ) ( t = 1.00 )</td>
<td>( i^* = 3.42 ) ( t = 1.00 )</td>
</tr>
<tr>
<td>( L_1 = 0.09 )</td>
<td>( P(TIE) = .173 ) ( P(TIE) = .169 )</td>
<td>( P(TIE) = .225 ) ( P(TIE) = .189 )</td>
</tr>
<tr>
<td>( L_1 = 0.06 )</td>
<td>( i^* = 2.35 ) ( t = 1.00 )</td>
<td>( i^* = 2.35 ) ( t = 1.00 )</td>
</tr>
<tr>
<td>( L_1 = 0.06 )</td>
<td>( P(TIE) = .235 ) ( P(TIE) = .217 )</td>
<td>( P(TIE) = .235 ) ( P(TIE) = .217 )</td>
</tr>
<tr>
<td>( L_1 = 0.09 )</td>
<td>( t = 2.58 ) ( t = 1.00 )</td>
<td>( t = 1.31 ) ( t = 1.00 )</td>
</tr>
<tr>
<td>( L_1 = 0.05 )</td>
<td>( P(TIE) = .346 ) ( P(TIE) = .315 )</td>
<td>( P(TIE) = .274 ) ( P(TIE) = .273 )</td>
</tr>
</tbody>
</table>

Table 3: Probability of a tie if \( T = 3.5 \) minutes.

...assumption will not matter that much. Unfortunately, this is not what we found based on the fixed-rate assumption (Table 2).

The interesting point to observe, as we compare Tables 1 and 2, is that the optimal times along each row in Table 1 are decreasing while the optimal times along each row in Table 2 are increasing. This means that under Erkut’s assumption (Table 1), a team that is losing by one goal should wait longer to pull the goalie the more goals their opponent can score. Under the constant rate assumption, the more goals the opponent scores, the sooner the trailing team should pull its goalie.

While the optimal time to pull the goalie is sensitive to various assumptions and estimates, the important issue is the probability of team 1 (which is trailing) obtaining a tie. We have computed this probability for the 25 pairs of scoring rates under both Erkut’s proportionality assumption and the fixed-rate assumption using the optimal times in Tables 1 and 2, and also using a (suboptimal) one minute time (as done by MW to represent coaching reality). We present these results in Table 3 for the two most sensitive cases. Probabilities under the proportionality assumption are more sensitive for high scoring teams, while probabilities for the fixed-rate assumption are more sensitive for low scoring teams. The extent of the sensitivity depends on the assumptions.

One conclusion of this work is that this model is not robust. We can, however, agree that, under either assumption, the latest a goalie should be pulled is 1.31 minutes. Since the conventional wisdom in the NHL is to pull the goalie with about one minute remaining in the game, we conclude that, under any set of assumptions, the coaches are too conservative and should be pulling goalies sooner than they do.

The most important point of this hockey model is that one must be careful in generalizing on the basis of insufficient data. Because one lacks information, one must make modeling assumptions, and these assumptions may affect the outcome significantly.

Acknowledgments

We are extremely grateful to the two referees for their suggestions and comments, some of which we have incorporated verbatim.

References


Morrison, D. G. 1976, “On the optimal time to pull the goalie: A Poisson model applied to
A comment from Donald G. Morrison, Graduate School of Business, Columbia University, New York, New York 10027.

Nydick and Weiss are gracious indeed for calling my initial effort "seminal" — but having just passed a milestone birthday, I wish they had not called this seminal work "ancient." In any event Nydick and Weiss's response to Erkut's reply to Morrison and Wheat's revision of the ancient and seminal Morrison should be the final chapter on our by now beloved old friend, The Goalie.

I only wish that the hockey establishment had taken the same interest in this problem as my fellow academics have. But there is hope on the horizon. Mitch Kupchak (the former NBA power forward, current assistant general manager of the Lakers and UCLA MBA) came to his alma mater last month to give a talk. Having taken the required OR course, Mitch quickly grasped the significance of this problem. With reprints of MW and Erkut in his hand, he returned to the Great Western LA Forum and passed these articles on to his Kings management colleagues down the hall. The Kings now have The Great One AND The Formula! It hardly seems fair — Gretzky and t* on the same team.